THE
MATHEMATICAL GAZETTE.

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.; PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.
PROF. E. T. WHITTAKER, M.A., F.R.S.

LONDON:
G. BELL AND SONS, LTD., PORTUGAL STREET, KINGSWAY,
AND BOMBAY.

VOL. VII. JANUARY, 1913. No. 102.

THE TEACHING OF LIMITS AND CONVERGENCE TO SCHOLARSHIP CANDIDATES. (*Continued.*)

8. The following application of Dedekind's principle is the most important one in practice.

If $s_1, s_2, s_3, \dots s_n, \dots$ be a sequence of positive rational numbers which steadily increase and never become greater than a given positive number K, then the given sequence approximates to a definite limit, K being rational.

We shall exclude the case when the members of the sequence become ultimately all equal (*e.g.* the decimal sequence approximating to $\sqrt{2\cdot25}$, viz. 1, 1·5, 1·50, 1·500, 1·5000, 1·50000, etc.).

We can then consider only the case when given s_n, we can always continue the sequence so far that we shall obtain members definitely greater than s_n.

Plainly, if s_n pass to the right of the point p, all subsequent members of the given sequence will lie to the right of p.

Let us consider, then, all the rational points lying in the interval $0, K+\epsilon$, where ϵ is an arbitrary positive rational number.

There are rational points (such as s_3) which the given sequence passes to the right of.

There are rational points (such as K) which the given sequence never passes to the left of.

We can therefore divide all the rational points in the interval $0, K+\epsilon$ into two classes:

Class (1). Rational points which the sequence passes to the right of, if n be chosen big enough.

Class (2). Rational points which the sequence never passes to the right of, however big n may be chosen.

Plainly, every member of Class (1) lies to the left of every member of Class (2).

Hence there is a schnitt separating the two classes.

9. *To shew that the schnitt s defined in Art.* 8 *satisfies the test of Art.* 1 *for being the limit of the sequence $s_1, s_2, s_3, \dots s_n, \dots$.*

We have to shew that, being given the positive rational number ϵ, however small, we can find a number n so big that s_n, s_{n+1}, etc., shall each differ from s by less than ϵ.

Apply the method of Art. 6, and let us find that the schnitt s lies between $k\epsilon$ and $(k+1)\epsilon$. Consider the rational number $k\epsilon$ (k a positive integer).

(1) Since $k\epsilon$ is a rational number lying to the left of s, we can find members of the given sequence that lie to the right of $k\epsilon$, *i.e.* between $k\epsilon$ and s (by the definition of s).

(2) Since $k\epsilon$ differs from $(k+1)\epsilon$ by ϵ,

$$\therefore \quad ks \text{ differs from } s \text{ by less than } \epsilon.$$

Hence all the members of the sequence lying between $k\epsilon$ and s differ from s by less than ϵ.

Corollary. *s is not greater than K.*

10. We can now apply direct methods to establishing the convergence of many important series and to calculate the error made by taking say s_n to represent s.

We shall assume that the Geometric Series has been investigated.

Let us denote the sequence $s_1, s_2, s_3, \ldots, s_n, \ldots$ by (s_n). We then have that the sequence $(1+x+x^2+\ldots+x^n)$ *continually increases and converges to the limit* $\dfrac{1}{1-x}$ *if x be numerically less than unity.*

11. *To prove that the sequence* $\left(\dfrac{1}{1^2}+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\ldots+\dfrac{1}{n^2}\right)$ *continually increases and converges to a limit which is* $\ngtr 2$.

Let
$$s_n=\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\therefore \; s_n<1+\frac{1}{1\cdot 2}+\ldots+\frac{1}{(n-1)n}\left(\text{writing in (1) } \frac{1}{(n-1)n} \text{ for } \frac{1}{n^2}\right),$$

$$\textit{i.e.} \; <1+\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)\ldots\left(\frac{1}{n-1}-\frac{1}{n}\right),$$

$$\textit{i.e.} \; <2-\frac{1}{n},$$

$$\textit{i.e.} \; <2.$$

$\therefore$ (s_n) is a continually increasing sequence, which never exceeds 2. Hence (s_n) defines a limit 2 (by Arts. 8 and 9).

To get a closer approximation to this limit, we can proceed as follows:

$$s_{n+p}=\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2}+\left\{\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\ldots+\frac{1}{(n+p)^2}\right\}$$

$$<\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2}+\left\{\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}+\ldots+\frac{1}{(n+p-1)(n+p)}\right\}$$

$$\textit{i.e.} \; <\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2}+\left\{\left(\frac{1}{n}-\frac{1}{n+1}\right)+\left(\frac{1}{n+1}-\frac{1}{n+2}\right)+\ldots+\left(\frac{1}{n+p-1}-\frac{1}{n+p}\right)\right\},$$

$$\textit{i.e.} \; <s_n+\left(\frac{1}{n}-\frac{1}{n+p}\right),$$

$$\textit{i.e.} \; <s_n+\frac{1}{n};$$

$$\therefore \; s_{n+p}, s_{n+p+1}, \ldots \text{ tends to a limit which is } \ngtr s_n+\frac{1}{n}.$$

Hence the limit s lies between s_n and $s_n+\dfrac{1}{n}$. *The error involved in taking s_n to represent s is less than* $\dfrac{1}{n}$.

If we take five terms, we have

$$s_4=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}=\frac{5269}{3600}=1\tfrac{1669}{3600},$$

and the error involved in taking $1\tfrac{1669}{3600}$ to represent s is less than $\frac{1}{5}$ or $\cdot 2$.

12. *Rapidity of Convergence.*

The idea of rapidity of convergence can now be placed before the student. In the above we have taken 5 terms, and are not sure even of the first decimal place.

13. We can also deal with the Exponential Series directly, and calculate the error.

Let $$s_n=1+\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}+\dots+\frac{1}{\underline{|n}}.$$

$$\therefore\ s_{n+p}=1+\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}+\dots+\frac{1}{\underline{|n}}+\frac{1}{\underline{|n+1}}+\dots+\frac{1}{\underline{|n+p}}$$

$$=1+\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}+\dots+\frac{1}{\underline{|n}}\left\{1+\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\dots+\frac{1}{(n+1)\dots(n+p)}\right\}$$

$$<1+\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}+\dots+\frac{1}{\underline{|n}}\left\{1+\frac{1}{n+1}+\frac{1}{(n+1)^2}+\dots+\frac{1}{(n+1)^p}\right\},$$

a fortiori

$$<1+\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}+\dots+\frac{1}{\underline{|n}}\cdot\frac{1}{1-\frac{1}{n+1}} \text{ (by the geometric series).}$$

$\therefore\ s_{n+p}, s_{n+p+1}\dots$ is a continually increasing sequence which never exceeds

$$1+\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}+\dots+\frac{1}{\underline{|n}}\cdot\frac{n+1}{n}.$$

$\therefore\ (s_{n+p})$ converges to a limit

$$\leqq 1+\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}+\dots+\frac{1}{\underline{|n}}\cdot\frac{n+1}{n} \text{ (in particular taking } n=1)\leqq 3.$$

This is a rapidly converging sequence, inasmuch as if we take 5 terms to represent s, the error is less than $\frac{1}{5\underline{|5}}$, *i.e.* $<\frac{1}{600}$.

14. This kind of treatment of the subject, which is at once arithmetical and direct, makes fairly easy to the student one of the undoubted stumbling-blocks in higher algebra. The great majority of the tests for convergence which we labour to instil into our pupils are of very little use for the cases of series which they are called upon to deal with. A very small percentage of them see what they are trying to do, and they apply the tabulated tests mechanically.

I have again to thank Dr. Charles M'Leod for criticising the paper while in manuscript form.

W. P. MILNE.

Clifton College.

A CASE OF THREE ROTATING LINES AND THE POINT "O." *(Continued.)*

IV.

15. Let X_1, Y_1, Z_1 be the second points in which AH, BH, CH meet the circle GH.

Then $\because$ $GX_1 \perp HX$, and hence $\parallel BC$,

$$\therefore \; X_1X = \tfrac{1}{3}AX\,;$$

$$\therefore \; \tan PX_1X = 3 \tan PAX = 3 \tan \theta,$$

$$i.e. \; \angle PX_1X = \phi.$$

But $OX_1H = OGH = \phi$; $\therefore$ OX_1P is a straight line.

Similarly for OY_1Q, OZ_1R.

The lines OP, OQ, OR therefore represent a system of three lines through the vertices of the $\triangle X_1Y_1Z_1$, rotating with equal but variable angular velocities ($\dot{\phi}$), and always meeting on the circumcircle. The $\triangle X_1Y_1Z_1$ is easily seen to be inversely similar to ABC.

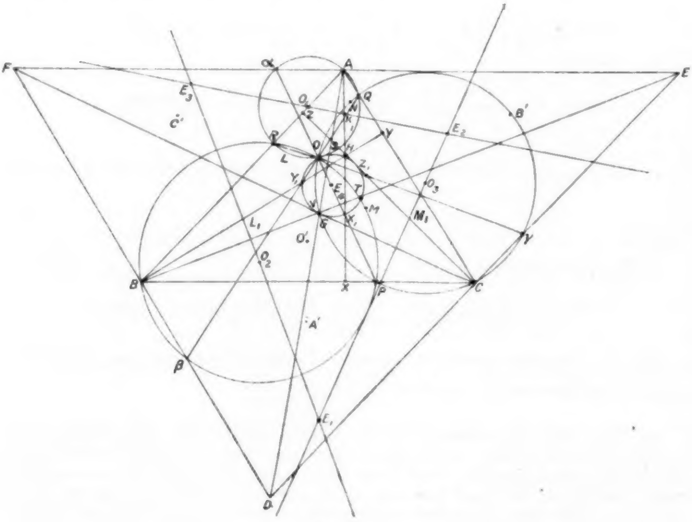


FIG. 2.

16. Let α, β, γ be the second points in which EF, FD, DE meet the circles AQR, etc.

Then $\angle A\alpha O = \pi - ASO = GSO = GX_1O$ or OPB, whence αOX_1 is a straight line.

Hence $P\alpha$, $Q\beta$, $R\gamma$ meet in O.

17. Since $\angle \beta O\gamma = QOR = \pi - A = \pi - D$, $\therefore$ $\beta O\gamma D$ is concyclic.

Hence $\odot^{\text{les}}$ $D\beta\gamma$, $E\gamma\alpha$, $F\alpha\beta$ meet in O.

18. Again, since $\angle SO\gamma = SAR = SD\gamma$,

$\therefore$ $\odot^{le}$ $D\beta\gamma$ passes through S.

The three circles $D\beta\gamma$, $E\gamma\alpha$, $F\alpha\beta$ therefore form a second triad of coaxal systems through the points D, S; E, T; F, V respectively.

19. If δ_1, δ_2, δ_3 are the diameters of these three particular circles, then, since $DO = \delta_1 \sin O\gamma D = \delta_1 \sin OPX = \delta_1 \cos\phi$, etc.,

$\therefore$ $\delta_1 : \delta_2 : \delta_3 = DO : EO : FO$.*

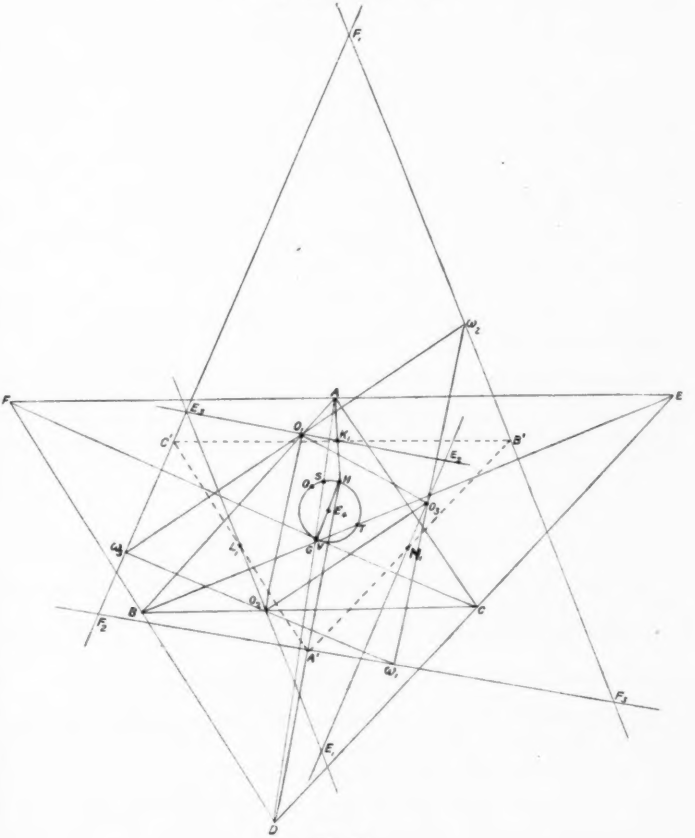


Fig. 3.

V.

20. Let ω_1, ω_2, ω_3 be the centres of these circles. (Fig. 3).
Then, since OS is a common chord of the circles O_1 and ω_1,

$\therefore$ $O_1\omega_1$ passes through E_4; similarly for $O_2\omega_2$, $O_3\omega_3$.

* This statement, like that of Art. 7, is true for the point O in the general case.

21. Let $F_1F_2F_3$ be the $\triangle$ formed by the lines of centres of the three systems (ω). Then, since F_2F_3 is the $\perp$ bisector of SD (Art. 18); since also $HS \parallel F_2F_3$, and since A' (the centre of the circle DBC) bisects HD,

$$\therefore\ A' \text{ lies on } F_2F_3.$$

In fact, A', B', C' are the initial positions of ω_1, ω_2, ω_3 respectively, the circles being in that case the three equal circles DBC, ECA, FAB.

The $\triangle^{\text{le}}$ $A'B'C'$ (which is identically equal to ABC) thus corresponds to $K_1L_1M_1$ of the first series O_1, O_2, O_3.

22. Since K_1, E_4, A' are mid-points, $A'K_1$ passes through E_4;

also $$E_4K_1 = \tfrac{1}{2}AG = \tfrac{1}{4}GD = \tfrac{1}{2}E_4A', \text{ etc.}$$

Hence the $\triangle^{\text{les}}$ $E_1E_2E_3$ and $F_1F_2F_3$, which are similar, have E_4 for their c.s.; that is to say, E_1F_1, E_2F_2, E_3F_3 meet in E_4 and trisect each other. The sides of the $\triangle^{\text{le}}$ $F_1F_2F_3$ are evidently twice those of $E_1E_2E_3$.

23. E_4 can be shewn to be the common symmedian point of the two triangles, for the trilinear coordinates of E_4 with respect to $F_1F_2F_3$ are as the sides of the $\triangle^{\text{le}}$.

E_4 is also of course the common centroid of $A'B'C'$ and $K_1L_1M_1$.

24. Let $O_1\omega_1$ meet O_2O_3 in M', so that $O_1E_4M'\omega_1$ are collinear (Art. 20). Then, since O_2O_3 is bisected by the median O_1M', and since

$$O_1M' = \tfrac{3}{2}O_1E_4 = \tfrac{1}{2}O_1\omega_1;$$

$$\therefore\ O_1O_2\omega_1O_3 \text{ is a } \square^{\text{m}}, \text{ and similarly for } O_2O_3\omega_2O_1 \text{ and } O_3O_1\omega_3O_2.$$

Hence $O_1O_2O_3$ is the median $\triangle$ of $\omega_1\omega_2\omega_3$.

25. It may be observed that, while the relation between the two triads $O_1O_2O_3$ and $\omega_1\omega_2\omega_3$ is the simple one, viz. that E_4 is their c.s., and that the scale of the latter is twice that of the former, the relations between the points PQR and $\alpha\beta\gamma$, and between the corresponding diameters $d_1d_2d_3$ and $\delta_1\delta_2\delta_3$, are quite different.

Thus we have $\dfrac{QR}{\beta\gamma} = \dfrac{d_1}{\delta_1} = \dfrac{AO}{DO}$ (Arts. 5 and 19); with similar relations for d_2, δ_2 and d_3, δ_3.

VI.

26. Various minor properties suggest themselves; we shall only mention two.

(1) Since $A\alpha = AX_1 \tan\phi = \dfrac{2}{3}AX\tan\phi = \dfrac{4}{3}\cdot\dfrac{\Delta}{a}\tan\phi$,

$$\therefore\ A\alpha\,.\,AF = B\beta\,.\,BD = C\gamma\,.\,CE = \frac{4}{3}\Delta\tan\phi.$$

Hence the tangents from A, B, C to the circles $F\alpha\beta$, $D\beta\gamma$, $E\gamma\alpha$ are equal.

(2) By Ptolemy's Theorem, we have

$$GS\,.\,TV + GT\,.\,VS + GV\,.\,ST = 0,$$

$$\text{i.e.}\quad GS\,.\,GA + GT\,.\,GB + GV\,.\,GC = 0 \text{ (Art. 3)},$$

or the algebraic sum of the squares of the tangents from G to the three circles O_1, O_2, O_3 is zero; *i.e.* the sum of the powers of G with respect to any triad is zero.

This property of G also holds good for the system $\omega_1\omega_2\omega_3$.

F. GLANVILLE TAYLOR.

REFORM OF MATHEMATICAL TEACHING IN GERMANY. (*Continued.*)

At the beginning of this century we find that there was a lively movement on foot to insist on the application of mathematics to actual problems occurring in daily life—a reaction from that purely formal mathematical teaching which for half a century had treated the subject-matter as of secondary importance so long as the logical training was achieved. This reaction became apparent amongst schoolmasters, especially in the meetings of the Förderungsverein, and it was even asserted by the extremists that Mathematics was nothing more than the handmaid of Science.

In 1894, at the meeting of this society in Wiesbaden a resolution was passed which asserted the desirability of employing occurrences of daily life and actual phenomena of nature in mathematical problems and exercises much more freely than was usual. At the same time, however, mathematics should not confine itself exclusively to such questions.

In the meanwhile there was at the Universities a tendency apparent to resuscitate the study of Applied Mathematics—the outcome of the increasing importance of technical and scientific studies. The moving spirit in this direction was Prof. *Klein* of Göttingen, who at the same time was taking the lead in the matter of the training of teachers. Klein has become the acknowledged leader of the reform movement in Germany, and he has published a mass of papers dealing with the teaching of mathematics in the school and at the University. To him are largely due the introduction of descriptive geometry as a special subject at the University and the organization of holiday courses for teachers.

Engineers also were severely criticising the teaching of Mathematics in School and University, and the combined agitation of Schoolmasters, University Professors and Engineers brought about notable reforms in the examination for fully qualified teachers, who, in 1898, were allowed to take Applied Mathematics as one of the three obligatory subjects permitted by the regulations.

In 1900 at the School Conference at Berlin, at which the 9-class Modern Schools were recognized as of equal standing with the classical Gymnasium, Klein and others urged the inclusion of the elements of the Calculus at any rate in the Modern Schools, but doubts were expressed as to the capacity of the boys to deal with the subject, and no reference was made to the matter in the Prussian syllabus of the following year, though there appeared a recommendation to make the problems dealt with in the mathematical class depend upon questions arising either in actual life or in the parallel course of Physics.

We now come to the period in which the introduction of Function into the curriculum became a more definite and practical question. In the Lehrplan of 1901 the idea was to be introduced at an earlier stage than formerly, but there was no recognition of it as the central notion of the whole mathematical course.

In 1902 there appeared the first school-book dealing with the function-idea which was to be treated in U II, but this text-book was not admitted into the schools until 1906. It must not, however, be supposed that the teaching did not take note of the regulation, for text-books are by no means a fair test of the work done in German classes, where the mathematical teaching is almost all oral, the whole class taking part in the solution of each question, while the working of sums in silence, which forms (or at any rate formed) so large a part of our mathematical teaching in England, is reduced to a minimum and confined almost exclusively to home-work.

The whole question was suddenly brought to a head by the action of the Scientists, who in 1901 put forth strong claims for Biology, which had been eliminated from the curriculum of the schools twenty years before. This was the opportunity for the Mathematicians to bestir themselves also, unless they were prepared to see the Scientists encroach upon the hours assigned to mathematics. Thus it came about that the representatives of Mathematics and Science became allies and set forth their views at the meeting of Natural Scientists at Breslau in 1904. At this meeting Klein claimed that "the function-idea graphically represented should form the central notion of mathematical teaching, and, as a natural consequence, the elements of the Calculus should be included in the curriculum of *all* 9-class schools," a procedure already proved to be possible in France, and incidentally bringing with it the advantage of disabusing the public mind of the idea that the Calculus was the stigma of the Realschule.

The meeting at Breslau received these suggestions with favour, and a Committee of twelve was appointed—the so-called "Breslauer Kommission" —to draw up definite reform proposals, which were laid before the general meeting of the Society in Meran in the following year (1905). These Meran proposals have become historic, and dealt with the teaching not only of Mathematics but also of Physics and Biology; but the special feature was the detailed Mathematical programme drawn up to test the practicability of the general proposals. In the construction of this programme the fundamental principles are: "to accommodate the course of teaching more than formerly to the natural process of mental development, to train as far as possible the faculty for contemplating surrounding natural phenomena from a mathematical point of view, and to make the pupil more and more consciously aware of the continuity of the subject as he passes from stage to stage,—a psychological, a utilitarian and a pedagogic principle."

From this may be inferred that, without prejudice to its value as formal training in logic, the chief objects of mathematical teaching must be—the development of judgments founded on observation and commonsense, and training in the habit of thinking of quantities as functions one of another.

In this "Meraner Lehrplan" (see *Mathematical Gazette*, December, 1911) the ideas of the Reformers were first crystallized into tangible form, and around their proposals has since centred the subsequent discussion of mathematical teaching.

The expression "funktionales Denken" or "functional thinking" was coined at Meran to describe the fundamental object of the new teaching, and it is difficult to decide whether the original word or its literal English translation is more objectionable from an aesthetic point of view. It has persisted, however, in Germany, owing to the timeliness of its original introduction.

According to the Breslau Committee, the emphasis laid upon the function-idea graphically represented is itself a preparation for the Calculus; but the extent to which this should be carried out gave rise to some difference of opinion: the large majority were in favour of introducing Calculus even in the Gymnasium, but a compromise was found by laying it down that the teaching in the Gymnasium should be carried "to the threshold of the Calculus."

These Meran proposals had such a widespread influence that during the very next year the Prussian Education authorities gave permission for five 9-class schools to make experiments in the direction indicated; two of these were Gymnasien, two Oberrealschulen and the fifth a Realgymnasium.

In response to a circular to all the 9-class schools of East and West Prussia it was found in 1908, that of the 26 schools from which answers were received (there are 38 such schools) 13 had introduced the function-idea and the Calculus. All of them agreed that the experiment was a success and were anxious to continue on the same lines.

After the publication of the Meran proposals the Breslau Committee continued the work of investigation and suggestion with a view to improvement in the teaching of Mathematics and Science, and published its final report at a general meeting of the Scientific Society at Dresden, dealing chiefly with the question of the training of teachers of those subjects, and advocating the grouping of Mathematics with Physics and Chemistry with Biology in the University courses. But with the completion of the task of the Breslau Committee a new and more widely representative standing committee was formed with the object of carrying out the reform proposals in detail. This new committee met in Cologne in January, 1908, under the name of the *Deutscher Ausschuss für mathematischen and naturwissenschaftlichen Unterricht* referred to in Germany under the arresting initials D.A.M.N.U. On this Committee of 36 members are represented all kinds of Mathematical and Scientific bodies, including Engineers and the Förderungsverein already mentioned. Among its original members we find Klein and Gutzmer—the latter as a representative of the Society of Scientists and Doctors and a well-known name in the reform movement.

In the course of its labours this new body dealt with many questions affecting all kinds of schools with which we are not concerned; but at its fourth session in March, 1910, the system of employing in the lowest classes of the 9-class schools masters who were not fully qualified academically—a system permitted by the Prussian authorities—was discussed and condemned.

At a meeting of the Scientists' Society at Salzburg, in September, 1909, *Hoppe*, a master at one of the Gymnasien at which an experiment in the new direction was being carried out in the three top classes only, reported as follows: "The boys have not only easily mastered the work, but with greater interest and less home-work drudgery than under the old system, and the result of examination encourages the hope that these ideas may be introduced generally into the curriculum of the classical schools."

At the same meeting, *Höfler* called attention to the influence of the Meran proposals upon the new Austrian Syllabus, which incorporated some of the new ideas, and thereby gave new impetus to the reform movement in Germany. A translation of this Austrian Syllabus has been issued by the Board of Education, and from it may be obtained a clear notion of the main tendencies of the reform movement.

We need not enter further into the discussion of these societies, but it may be interesting to note the attitude of the schoolmasters as a whole. In each of the thirteen provinces of Prussia the headmasters of all higher schools hold official meetings every four years to discuss any burning questions of educational interest. Among headmasters the mathematicians are very few—in 1910, 108 out of 691, or about 16 per cent.—but the general trend of opinion in those meetings in which the Meran proposals were discussed has been decidedly friendly. Among the assistant masters, too, the reform proposals were welcomed, as may be gathered from the discussions which arose in the so-called Philologen- vereine, or meetings of fully qualified teachers, which take place every year in each province.

At a meeting of the Mathematical Society of Berlin, however, in 1909, one speaker showed a tendency to criticise the new movement; objection was taken to the early introduction of the function-idea, which, said he, was dealt with early enough in connection with Trigonometry. On the other hand, a second speaker maintained that in the matter of geometry every intelligent teacher had been carrying out the new ideas for years!

In the holiday-courses for teachers, too, the subject has been continually under discussion, and Klein has made specially noteworthy contributions, many of which have been published.

Any further account of the progress in the reform of Mathematical teaching in Germany necessarily brings us to the International Committee for the Teaching of Mathematics, which is the outcome of the efforts of the teachers

in all civilized countries to arrive at some conclusion as to the proper reform in the teaching of the subject. In response to a suggestion on the part of Prof. Smith of New York, at the International Congress of Mathematicians in Rome, 1908, this Commission—known in Germany by its initials M.U.K.—was appointed to study and compare the present tendencies in the teaching of mathematics of different countries, to investigate the history of their origin and development, and to report at the next meeting of the Congress in Cambridge in 1912. Klein was appointed chairman of the Committee which drew up the Constitution of the Commission, and sub-committees were formed in each country.

The German sub-committee has drawn up a series of reports in the methodical Teutonic fashion, and this paper is to a large extent a *résumé* of one of these.

Evidence as to the progress of the reform movement may be obtained from the books in use in the schools, but I would again insist on the point that the use of any book is not necessarily an indication of the lines on which the subject is taught. The number of school-books which include work on reform lines is continually growing and is fairly considerable already. Of the text-books, as distinct from mere collections of problems, two categories may be distinguished: the new editions of old books which introduce the function-idea and its representation graphically in a special appendix, without giving any indication as to its bearing on the former subject-matter, and the newer books which introduce the notion in the early stages and endeavour to carry out the suggested reforms throughout the course. In these books the fundamental notion has to some extent modified the accepted order of dealing with the subject-matter. In the text-book of *Schwab* and *Lesser* simple graphs accompany the first exercises in Algebra, and play a leading part in the solution of linear equations, and the consideration of functions leads in the Oberstufe to the general investigation of rational algebraical function. In like manner several collections of problems have either been rewritten or newly published, with the object of appealing to the progressive, while at the same time retaining popularity among the more conservative teachers. And, lastly, we find a number of books whose object is not to treat the whole school course of mathematics, but to supply a methodical treatment of the function-idea in conjunction with books already in use.

In conclusion I may add that, in correspondence with Herr Schimmack (the author of the pamphlet from which this paper chiefly draws its information and Klein's principal lieutenant in the movement for reform), I learn that the tendency to introduce the function-idea from the very beginning of the school course is apparently making further progress, though there is great variety in the methods of teachers even in the same school. There is, however, little doubt that in the next official programme a definite position will be taken up in the direction of reform; modifications of this kind have already been made in the programme for Girls' and Higher Elementary Schools. There is no sign of serious opposition or reaction, though some notable teachers stand aside, and many adopt a hesitating but not hostile attitude.

E. A. PRICE.

MATHEMATICAL NOTES.

382. [**V. a; R. 6.**] *The Unit of Momentum.*

Doubtless all teachers of experience are agreed that in dealing with the measurable quantities considered in Mathematics and Physics, there is great difficulty in giving to students a clear idea of the quantities measured unless a definite name be given to the unit in terms of which measurement is made.

Until the word "radian" was introduced, circular measure was a stumbling block to all beginners in Trigonometry; the sentences "the circular measure of this angle is 1·7" and "arc over radius equals 1·7" make little impression on the untutored mind; but the sentence "this angle contains 1·7 radians—the radian being rather more than 57°" is as easy of comprehension as "the value of this is $2\frac{1}{3}$ guineas—the guinea being 21 shillings." So in Electricity, without the names Ampere, Volt, Ohm, etc., the learner would be—as he was thirty years ago—lost in hopeless vagueness.

In Mechanics we have given names to almost all the Units; why not give one to the Unit of Momentum? I venture to suggest for general use the names which I have myself made use of for many years, and which I have found helpful in inducing clear thinking on the part of my students; they are analogous to the names of the units of work. Thus:

the *work* done by a force 1 lb. acting for 1 ft. is one *ft.-lb.*;
the *momentum* produced by a force 1 lb. acting for 1 sec. is one *sec.-lb.*;

the *work* done by a force 1 poundal acting for 1 ft. is one *ft.-poundal*;
the *momentum* produced by a force 1 poundal acting for 1 sec. is one *sec.-poundal*;

the *work* done by a force 1 dyne acting for 1 cm. is one *cm.-dyne* (erg);
the *momentum* produced by a force 1 dyne acting for 1 sec. is one *sec.-dyne.*

Similarly we may speak of a sec.-ton and a sec.-tondal.

The advantage of these names is that they emphasize the essential distinction between Momentum and Kinetic Energy, viz. that one measures what I may call the time-effect of a force and the other its space-effect; or, to put it differently, when we know the momentum of a moving body, we know *how long* a given force must act to produce the motion; and when we know its Kinetic Energy, we know *how far* the force must act. This point of view seems to me far more desirable than the suggestion to the beginner that Momentum is mv or "mass × velocity," such a suggestion he must either receive in deadly apathy, or he must worry his brain with the impossible task of trying to conceive how mass can be *multiplied by* velocity; his teacher may try to slur over the difficulty for him by a word-juggling substitution, and say "momentum is the *product* of mass and velocity"; but then—what is a product? In its mathematical sense, the word product means the result of multiplication, and in its non-mathematical sense it means anything produced by or resulting from certain other things; in this latter sense, Kinetic Energy is just as much the product of mass and velocity as momentum is. So also the time-honoured but vague phrase "quantity of motion" could be used to express Kinetic Energy with quite as much aptness as to express Momentum.

Using these names, we would teach:—"the Momentum of m lb. moving at v f.s. is mv sec.-poundals or mv/g sec.-lbs.; its K.E. is $\frac{1}{2}mv^2$ ft.-poundals or $\frac{1}{2}mv^2/g$ ft.-lbs."

"The momentum of m grammes moving at v c.s. is mv sec.-dynes; its K.E. is $\frac{1}{2}mv^2$ ergs (cm.-dynes)."

F. R. BARRELL.

The University, Bristol.

383. [L[1]. 1. a.] *On an interpretation of the general equation of a conic, and a certain connected theorem.*

Let $T(x, y)$ be any point in the plane of the conic

$$aX^2+2hXY+bY^2+2gX+2fY+c=0,$$

where X, Y are current coordinates; and draw $TPEP'$ the secant through T and the centre E of the conic. Transferring to E as origin, the equation

becomes $aX^2+2hXY+bY^2+c'=0$, where $c'=\Delta/C$ and $C=ab-h^2$; and the square of the semidiameter making an angle θ with the axis of x

$$=(-c')/\{a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta\}.$$

For EP we have $\cos\theta=\{x-G/C\}/ET$ and $\sin\theta=\{y-F/C\}/ET$.

Thus

$$ET^2/EP^2-1=\{a(x-G/C)^2+2h(x-G/C)(y-F/C)+b(y-F/C)^2+c'\}/(-c')$$
$$=ax^2+2hxy+by^2+2gx+2fy+c)/(-c')$$

and $ax^2+2hxy+by^2+2gx+2fy+c \propto TP.TP'/EP^2$ or $\propto \{ET^2/EP^2-1\}$.

If T lies on an asymptote, ET is finite and EP infinite. Thus $ET/EP=0$, and the equation of the asymptotes is $ax^2+2hxy+by^2+2gx+2fy+c-c'=0$.

The above leads to the theorem (suggested by the Rev. J. J. Milne):— If $S=0$, $S'=0$ are the equations of two conics, the locus of T, which moves in such a manner that a tangent therefrom to the first conic divided by the parallel diameter of that conic has always a constant ratio to the tangent therefrom to the second conic divided by its parallel diameter, is a conic $S-kS'=0$ passing through the intersections of $S=0$ and $S'=0$.

R. F. Davis.

Mr. Milne's theorem is as follows: A and B are two conics of a pencil, C, C' their centres. T is any point on M, a third conic of the pencil. TC meets A in a, a'.

T' is any other point on M, and TT' meets A in α, α'. Ct is the semi-diameter of A, parallel to TT'.

Then $Ta.Ta':T\alpha.T\alpha'=Ca^2:Ct^2$.

Let $T'C$ meet A in a_1, a'_1.

Then $T'a_1.T'a'_1:T'\alpha.T'\alpha'=Ca_1^2:Ct'^2$, and $Ct=Ct'$.

$$\therefore\ \frac{Ta.Ta'}{Ca^2}:\frac{T'a_1.T'a'_1}{Ca_1^2}=\frac{T\alpha.T\alpha'}{T'\alpha.T'\alpha'}. \quad\text{.........................(1)}$$

Using corresponding letters for the conic B, we have

$$\frac{Tb.Tb'}{c'b^2}:\frac{T'b_1.T'b'_1}{C'b_1^2}=\frac{T\beta.T\beta'}{T'\beta.T'\beta'}. \quad\text{.........................(2)}$$

Now, by Desargues' Theorem, $(TT',\ \alpha\alpha',\ \beta\beta')$ is a range in involution.

$$\therefore\ (TT'\alpha\beta)=(T'T\alpha'\beta');\quad \therefore\ \frac{T\alpha.T\alpha'}{T\beta.T\beta'}=\frac{T'\alpha.T'\alpha'}{T'\beta.T'\beta'}.$$

$\therefore$ from (1) and (2) we have

$$\frac{Ta.Ta'}{Ca^2}:\frac{Tb.Tb'}{C'b^2}=\frac{T'a_1.T'a'_1}{Ca_1^2}:\frac{T'b_1.T'b'_1}{C'b_1^2}.$$

Now T, T' are any two points on M;

$$\therefore\ \frac{Ta.Ta'}{Ca^2}:\frac{Tb.Tb'}{C'b^2}=\text{const.}$$

Conversely, we have the theorem:

If T is a point such that if through C, C', the centres of two conics A, B, we draw TC, TC' meeting A, B respectively in a, a' and b, b',

$$\frac{Ta.Ta'}{Ca^2}:\frac{Tb.Tb'}{C'b^2}=\text{const.},$$

the locus of T is a conic of the pencil determined by A and B.

If TP is a tangent to A and Cp the semi-diameter of A, parallel to TP, with corresponding letters for B,

$$\frac{Ta.Ta'}{Ca^2}=\frac{TP^2}{CP^2},\quad \frac{Tb.Tb'}{C'b^2}=\frac{TP'^2}{C'p'^2}.$$

$\therefore \dfrac{TP}{Cp} : \dfrac{TP'}{C'p'} = \text{const.}$, and we obtain the theorem given at the end of Mr. Davis' note.

For a geometrical proof of the corresponding theorem for three coaxial circles, see Townsend, *Modern Geometry*, Vol. I., Art. 192, Cor. 1.

JOHN J. MILNE.

384. [V. 1. a.] My friend Mr. Rose-Innes has communicated to me a most interesting and important criticism of Euclid's proof of the 6th Proposition of the Tenth Book of his elements.

The proposition in question is this: If two magnitudes have to one another the ratio which a number has to a number, the magnitudes will be commensurable.

As the subject is connected with the matter dealt with in my Presidential Address, I send it to you in the hope that you may be able to print it.

Euclid assumes in his proof that any magnitude can be divided into as many equal parts as there are units in a given number.

Mr. Rose-Innes points out that this assumption is not necessary.

It will be seen that his proof given below depends on the process of finding the Greatest Common Measure, and does not assume the divisibility of a magnitude into any number of equal parts.

Consequently the Euclidean Theory of Proportion *may be freed from its dependence on the above assumption.*

Though the arrangement of the argument, as suggested by me in the July and October numbers of the *Gazette* is (I venture to think) better adapted than Euclid's for application to numerical calculations, yet it is essentially dependent on the assumption referred to.

I desire to take this opportunity of correcting an oversight in my Presidential Address. The 5th, 6th and 7th Propositions of the Tenth Book should have been associated with the 4th and not with the 2nd of the groups in which I have arranged the propositions of the 5th Book.

I am indebted to Mr. Rose-Innes for this correction. M. J. M. HILL.

His proof is as follows:

If two magnitudes have to one another the ratio which a number has to a number, the magnitudes will be commensurable.

Suppose that $X : Y = a : b,$

then $rX : sY = ra : sb.$

Taking $r = b, \quad s = a,$

$$bX : aY = ba : ab,$$

but $ba = ab ; \quad \therefore bX = aY.$ ....(I)

Now, if $X > Y$, then $b < a$.

Perform on X, Y the process of finding the greatest common measure.

Let
$$\left.\begin{aligned} X &= q_1 Y + R_1 \\ Y &= q_2 R_1 + R_2 \\ R_1 &= q_3 R_2 + R_3 \\ &\cdots\cdots\cdots\cdots \\ R_{n-2} &= q_n R_{n-1} + R_n \end{aligned}\right\} \quad \text{..............................(II)}$$

Then, from (I), $b(q_1 Y + R_1) = aY ;$

$\therefore (a - bq_1) Y = bR_1.$ ....(III)

Put $a - bq_1 = r_1 ;$ ....(IV)

$\therefore r_1 Y = bR_1 ;$ ....(V)

$\therefore r_1(q_2 R_1 + R_2) = bR_1 ;$

$$\therefore\ (b-r_1q_2)R_1=r_1R_2,$$

or, putting
$$b-r_1q_2=r_2\ ; \qquad \text{(VI)}$$
$$\therefore\ r_2R_1=r_1R_2. \qquad \text{(VII)}$$

Proceeding thus, we get ultimately

$$r_nR_{n-1}=r_{n-1}R_n. \qquad \text{(VIII)}$$

Now the equations
$$a-bq_1=r_1,$$
$$b-r_1q_2=r_2, \text{ etc.},$$

show that $r_1, r_2, \dots r_n$ are the remainders in the process of finding the greatest common measure of the positive integers a, b.

Hence we must come at last to equations of the form

$$r_{n-2}=q_nr_{n-1}+r_n,$$
$$r_{n-1}=q_{n+1}r_n,$$

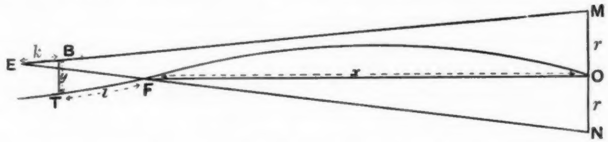
the remainder r_{n+1} being zero, because a, b being positive integers have a common measure.

But
$$r_nR_{n-1}=r_{n-1}R_n\ ;$$
$$\therefore\ R_{n-1}=q_{n+1}R_n.$$

$\therefore$ R_n measures R_{n-1}, and therefore R_{n-2}, $R_{n-3}, \dots R_1$, Y, X. Hence X and Y have R_n for a common measure.

385. [R. 7. b. γ. S. 6.] *The Ommundsen Rifle Sight.*

The essential feature of the sight is that it is so arranged as to require no alteration, whatever the range may be. The man fired at is callipered between the fore and back-sight, so that the further he is from the firer the smaller he appears and the lower is the trajectory obtained.



The target must be assumed to be of constant height, *i.e.* that of a man standing.

Suppose MON represents the target, of height $2r$.

Then if F be the fore-sight and TB the back-sight of a rifle fired by a man whose eye is at E, EBM and EFN will be straight lines.

Let $TF=l$, $EB=k$, $FO=x$; then k, l, r and TB are assumed to remain constant, whatever the length (x) of FO.

Now if BT, the height of the back-sight, $=y$, and a is the inclination of the barrel, then

$$y=\frac{kr}{x}+\frac{(l+k)r}{x}+la$$

$$=(l+2k)\frac{r}{x}+\frac{lxg}{2v^2}, \text{ since, when } a \text{ is small, } x=\frac{2v^2a}{g},$$

v being the muzzle velocity of the bullet.

Now, when x varies, y will not remain accurately constant. To find the amount of error in this respect, suppose:

$l=33\cdot\dot{3}$ ins.; $g=32$ ft. per sec.^{2}; $k=8\cdot\dot{3}$ ins.;

$v=2000$ ft. per sec.; $x=300n$ ft.; $r=3$ ft.

Then $$y=(33\dot{3}+16\dot{6})\frac{3}{x}+\left(\frac{33\dot{3}\times 32x}{2\times 4000000}\right) \text{ ins.}$$
$$=\frac{1}{2n}+\frac{n}{25}.$$

Hence y will be a minimum when $n=\frac{5}{\sqrt{2}}=3\cdot54$, and then $y=\cdot28$. in.

For $n=1, 2, 2\cdot5, 5, 10, 20$, we find $y=\cdot54, \cdot33, \cdot3, \cdot3, \cdot45, \cdot83$ respectively.

Hence y is not strictly constant, but the variation is extremely small. For ranges between 200 yds. and 1000 yds. the minimum value of y is ·28 in., and the maximum value is ·45. Hence the variation is only ·17 in., while from 200 yds. to 2000 yds. the variation is ·55 in. With the old pattern sight the variation from 500 to 1100 is ·5 in. and from 600 to 1800 1·5 ins.

The importance or insignificance of these variations will best be seen by observing the error involved when they are neglected. Let us assume that y is kept fixed and $=\cdot28$ in.

Then if the man is assumed to be at a distance $x_1=300n_1$ ft.,

$$y=(l+2k)\frac{r}{x_1}+la \quad \text{or} \quad \cdot28=\frac{50\times 3}{x_1}+33\cdot3a;$$
$$\therefore \ \cdot28=\frac{1}{2n_1}+\frac{100a}{3} \quad \text{if } x_1=300n_1;$$
$$\therefore \ a=\frac{3}{100}\left\{\cdot28-\frac{1}{2n_1}\right\}=\cdot0084-\frac{\cdot015}{n_1}.$$

Also the horizontal distance travelled by the shot is $\frac{2v^2a}{g}$ ft. $=300n_2$ ft. suppose.

Then $n_2=\frac{10000}{12}a$. But when the shot crosses the horizontal line through the muzzle it is at a height of 3 ft. Hence, before it hits the ground, it travels a distance $v\sqrt{\frac{6}{g}}=500\sqrt{3}$ ft. $=\frac{5}{\sqrt{3}}100$ yds.

Therefore, total distance travelled by the shot before it strikes the ground is $\left\{\frac{10000a}{12}+\frac{5}{\sqrt{3}}\right\}100$ yds. Hence the bullet will strike the man if

$$n_1<\frac{10000a}{12}+\frac{5}{\sqrt{3}}, \quad \text{and} \quad \frac{10000a}{12}=7-\frac{12\cdot5}{n_1}.$$

Therefore the man will be hit if $n_1<7-\frac{12\cdot5}{n_1}+2\cdot88$.

Now, if $n_1=9\cdot88-\frac{12\cdot5}{n_1}$, $n_1^2-9\cdot88n_1+12\cdot5=0$, whence $n_1=8\cdot4$.

Hence the man will be hit without any alteration of the sight, so long as he is not more than 840 yds. from the firer.

It has already been seen that for ranges below 200 yds. the arrangement is not accurate, so that the limits of utility are practically those required in modern warfare, *i.e.* from 200 to 800 yds. H. C. Frampton.

386. [D. 6. b; V. a.] The following treatment of the fundamental exponential limit, $\underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)^n$, is exceedingly simple and may be new to some readers of the *Gazette*.

If $a_1, a_2, \ldots, a_n$ be any number of positive proper fractions,

$$1>(1-a_1)(1-a_2)\ldots(1-a_n)>1-(a_1+a_2+\ldots+a_n). \qquad \text{............(1)}$$

First taking n a positive integer,

$$\left(1+\frac{1}{n}\right)^n = 1+\frac{1}{\underline{|1}}+\frac{1-\frac{1}{n}}{\underline{|2}}+\ldots+\frac{\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{n-1}{n}\right)}{\underline{|n}}. \quad \ldots\ldots\ldots\ldots(2)$$

Using (1), we easily find

$$\frac{1}{\underline{|r}} > \frac{1}{\underline{|r}}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{r-1}{n}\right) > \frac{1}{\underline{|r}} - \frac{1}{2n}\cdot\frac{1}{\underline{|r-2}}.$$

Taking the expansion (2) term by term and applying this inequality,

$$\begin{array}{lll}
1 = & 1 & = 1, \\
\frac{1}{\underline{|1}} = & \frac{1}{\underline{|1}} & = \frac{1}{\underline{|1}}, \\
\frac{1}{\underline{|2}} > & \frac{1}{\underline{|2}}\left(1-\frac{1}{n}\right) & = \frac{1}{\underline{|2}} - \frac{1}{2n}, \\
\vdots & & \vdots \\
\frac{1}{\underline{|r}} > & \frac{1}{\underline{|r}}\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{r-1}{n}\right) & > \frac{1}{\underline{|r}} - \frac{1}{2n}\cdot\frac{1}{\underline{|r-2}}, \\
\vdots & & \vdots \\
\frac{1}{\underline{|n}} > & \frac{1}{\underline{|n}}\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{n-1}{n}\right) & > \frac{1}{\underline{|n}} - \frac{1}{2n}\cdot\frac{1}{\underline{|n-2}};
\end{array}$$

adding and writing $\qquad S_n = 1+\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}+\ldots+\frac{1}{\underline{|n}}$,

we get
$$S_n > \left(1+\frac{1}{n}\right)^n > S_n - \frac{1}{2n}\left(1+\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}+\ldots+\frac{1}{\underline{|n-2}}\right)$$
$$> S_n\left(1-\frac{1}{2n}\right);$$

dividing by S_n,
$$1 > \frac{\left(1+\frac{1}{n}\right)^n}{S_n} > 1-\frac{1}{2n},$$

which shows that
$$\mathrm{Lt}\left(1+\frac{1}{n}\right)^n = \mathrm{Lt}\, S_n = e.$$

If n is not integral, let $m < n < m+1$, where m is an integer. Then evidently

$$\left(1+\frac{1}{m+1}\right)^m < \left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{m}\right)^{m+1}, \text{ etc.}$$

$\left(1-\frac{1}{n}\right)^{-n}$ may be treated in a similar manner, or, since

$$\frac{\left(1+\frac{1}{n}\right)^n}{\left(1-\frac{1}{n}\right)^{-n}} = \left(1-\frac{1}{n^2}\right)^n,$$

we have by (1),
$$1 > \frac{\left(1+\frac{1}{n}\right)^n}{\left(1-\frac{1}{n}\right)^{-n}} > 1-\frac{1}{n}.$$

Thus
$$\mathrm{Lt}\left(1-\frac{1}{n}\right)^{-n} = \mathrm{Lt}\left(1+\frac{1}{n}\right)^n.$$

Of course the same arguments may be used in dealing with the more general limit of $\left(1+\frac{x}{n}\right)^n$ or $\left(1+\frac{1}{n}\right)^{nx}$.

J. LISTER.

REVIEWS.

The XY. Edited by NAGAZAWA KAMENOSUKE. Vol. VIII. No. 6. (Published by the XY House: printed in the Japanese language and character.)

This is a magazine of elementary mathematics. It contains solutions to questions set in previous numbers, questions for solution, anecdotes (movement of the earth, pointing drawing pens, etc.), and correspondence. It is apparently run by private enterprise. The questions and solutions are quite elementary. A lecture on Physics (under the title Engineering) deals with the measurement of time, longitude, and the calendar. Then follows an account of various methods of measuring distance: the note on estimation by eye is of some interest, considerable attention being paid to it in the Army. "People with middling sight can see from 3500 to 4000 metres in clear weather and from 800 to 1200 in rainy weather. Capability for measuring distance depends on experience, but error ought not to exceed $\frac{1}{4}$th distance measured." Then follows a list of objects suggested for practical experiment.

Some of the questions for solution have a touch of local colour. "Percentage of salt in sea water" has a real meaning to people whose salt comes from pans. "Cubic contents of a moat" would be well understood in Tōkyō or any other city.

In a note on the properties of a plane triangle, the following is set:

ABC is a triangle. Three points D, E, F are taken in the sides, so that

$$\frac{BD}{a}=\frac{CE}{b}=\frac{AF}{c}=k, \text{ say.}$$

If BE and CF; CF and AD; AD and BE meet in α, β, γ respectively, show that the centroid of a triangle ABC coincides with that of $\alpha\beta\gamma$. A proof is given which is analytical in disguise.

Nunelaus' (?) Theorem is quoted to the effect that

$$A\gamma : \gamma D = BC \, . \, EA : DB \, . \, CE.$$

The translator must plead guilty of ignorance in regard to Nunelaus (?) and his Theorem. G. H. RAYMENT.

Die Integralgleichungen und ihre Anwendungen. By ADOLPH KNESER. Pp. vi, 243. 1911. (Vieweg, Braunschweig.)

An integral equation is an equation involving signs of integration just as a differential equation involves signs of differentiation. Thus

$$y=1+\int xy\,dx \text{ is the same as } \frac{dy}{dx}=xy. \quad y=1 \text{ when } x=0.$$

Like so many other developments in mathematics, integral equations spring from a paper of Abel's. In considering the problem (*Œuvres*, vol. i. p. 11) of finding the curve down which a particle would fall from rest under gravity through a height y in a time equal to an assigned function of y, say $\psi(y)$, we are led (with $2g=1$) to the equation

$$\psi(a)=\int_0^a \frac{ds}{\sqrt{(a-y)}},$$

which Abel solves by assuming $s=\Sigma A_m y^m$, integrating by Gamma functions, replacing $\psi(a)$ by a power series, and equating coefficients. Professor Böcher's criticism on Abel's solution, "Abel omits the essential step of proving that the equation has a solution," indicates vividly the difference in attitude of mind between the physicist and the mathematician.

The subject has developed immensely in recent years, and some account of modern progress will be found in Bateman, *Proc. L.M.S.*, vol. iv. (new series), pp. 90-115; in Böcher, *Cambridge Tracts*, "Integral Equations"; and in *L'Equation de Fredholm et ses applications*, by Heywood and Frechet.

Taking
$$\phi(s)-\lambda\int_a^b k(s\,.\,t)\,\phi(t)\,dt=f(s)$$

as the typical equation, k and f being known functions and ϕ the function to be determined, two main methods of solution have been developed by Neumann, Volterra and Fredholm.

The peculiar feature which in the opinion of some mathematicians renders the theory likely to prove of practical value in physics is that a single integral equation is equivalent to a differential equation together with certain equations of condition.

Professor Kneser divides his treatise into six sections dealing with the application of integral equations to problems of the linear conduction of heat, the oscillations of chains and rods, the relation to problems of expansion and Bessel-Legendre functions, two and three-dimensional heat problems, existence theorems, and Fredholm's determinant solutions. Of the authors above mentioned, Professor Kneser leads most directly to physical applications. A development in accordance with the lines of his treatment has just been published by Professor Carslaw in the *Proceedings of the Edinburgh Mathematical Society*. C. S. J.

The Pell Equation. By Ed. E. Whitford. Pp. iv+193. $1 post free. 1912. (College of City of New York.)

This book contains an account of the Pellian equations $x^2-Ay^2=\pm 1$, and of some of the equations connected therewith. The text (102 pp.) is chiefly historical; there are 49 pages of bibliography, and 38 pages of (new) Tables.

The name "Pellian," by which this equation is now known, is shown (pp. 1, 2) to be a misnomer, as John Pell seems to have done little work on it. Approximations to square roots ($\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc.), which are in fact solutions of the equation (with $A=2$, 3, 5, etc.), are shown to have been well known to the early Hindus, Greeks, and Arabs: an interesting account is given (pp. 3-45) of their very considerable work hereon. This is followed by a discussion (pp. 47-76) of the contributions of Euler, Fermat, Brouncker and Lagrange to the complete solution of the equations. The first complete practical solution is ascribed to Brouncker (pp. 49-52), and the first valid complete proof of the solvability for every positive non-square integer A is ascribed to Lagrange (pp. 71, 72).

The text ends with a short account (pp. 77-92) of the contributions and developments by Legendre, Jacobi, Dirichlet, and Kronecker. Thus, in the short compass of 94 pages is presented a masterly view of the history of this most interesting equation from very early days right down to the present time: the discussion is supported by copious notes and references.

A short account of previous Tables of solution (Euler's, Legendre's, Degen's, Bickmore's, and others) is given on pages 95-97, and is followed by (new) Tables of solutions of both equations $x^2-Ay^2=\pm 1$ (when possible) for all positive non-square values of A from 1501 to 1700 (pp. 102-112).

Also, in an Appendix (pp. 162-190) is given the development of $\sqrt{A}$ in a continued fraction for $A=1501$ to 2012 (on the pattern of Degen's and Bickmore's Tables). These Tables are a most valuable addition to the previous Tables (being a continuation of Bickmore's, which stop at $A=1500$): the typography is by no means convenient, large numbers of 54 figures being printed in one line continuously (without any break); these are quite difficult to read.

The literature of this equation is very extensive. A most useful bibliography of over 300 entries of works from 1798 to 1910 is given on pages 113-161. This list includes many Questions from the *Educational Times* and *L'Intermédiaire des Mathématiciens*: some of these are trifling and hardly worth insertion. It is extremely difficult to make any such list exhaustive: several papers by Ed. Lucas are missing from the list—some of them really important—*e.g.* his memoirs, *Théorie des Fonctions Numériques simplement périodiques* in the *American Journal of Mathematics, Pure and Applied*, vol. i., 1878.

In this bibliography references are given to solutions for various special values of $A>2000$, scattered over a number of papers: it would be useful to collect these and publish them together along with any such work as the present.

Tables de Logarithmes à 3 Quatrades et Nombres correspondants avec 12-13 chiffres. Système normal du Dr. Auguste Guillemin. 22 pages Introduction, 100 pages Tables, 24 pages Supplement. 1912. (Paris.)

This work aims at giving, in a Table of only 100 pages, the means of finding quickly by purely logarithmic work (*i.e.* by simple additions and subtractions only) 12 or 13 figure numbers and logarithms of given 12 or 13 figure logarithms and numbers respectively. The arrangement is very ingenious. Let the

decimal part (mantissa) of the 12 figure logarithm of a number X consist of three sets (Q_1, Q_2, Q_3) of four digits each (hence styled "Quatrads"), so that $\log X = Q_1 + Q_2 + Q_3$. And let N, $(1+\alpha)$, $(1+\beta)$ be the numbers whose logs are Q_1, Q_2, Q_3 respectively, so that $X = N(1+\alpha)(1+\beta) = N + N\alpha + N\beta + N\alpha\beta$.

The Table (of 100 pages) gives the number N (to 12 or 13 figures), and the value of $\log \alpha$ (to 8 figures) for every value of Q from 1 to 9999. The column of $\log \alpha$ may be taken as one of $\log \beta$ also. Here N is a rough approximation to X, $N\alpha$ is a "first correction," $N\beta$, $N\alpha\beta$ are successive "corrections" (rapidly decreasing in magnitude): their values can be quickly found by pure logarithmic work (*i.e.* by simple additions and subtractions) from the Tables themselves. "Differences" are also given in the columns of N and $\log \alpha$ for correcting the last figures, and the values of N and of $\log \alpha$, and the differences are also *marked* (Fr. *pointé*) with a set of marks whether the next figure is nearest to 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$. The mode of use is quite simple, and could be readily acquired with a little patience.

ALLAN CUNNINGHAM, Lt.-Col., late R.E.

Les Anaglyphes géométriques. By H. VUIBERT. 1s. 6d. 1912. (Vuibert, Paris.)

One of the most striking features of the Exhibition of Mathematical Instruments, Models, etc., held at Cambridge in connexion with the International Congress of Mathematicians, was the beautiful series of "Anaglyphes," or "Plastographs," of various mathematical figures, in which a stereoscopic effect was produced by viewing bi-coloured diagrams through absorption glasses. The general method was discovered by W. Rollman and described by him in *Poggendorff*'s *Annalen* for 1853 (90, p. 187), as "Farbensteoroscop bestehend aus eine farbigen Doppelzeichnung und zwei gefärbten Gläsern." His method is thus summarised by Dr. von Rohr in his *Binoculare Instrumente*: "Die zusammengehörigen Halbbilder, sich gegenseitig durchschneidend mit zwei verschiedenen Farben auf einen weissen Grund gezeichnet. Vor jedes der beiden Augen wurde nun ein Farbglas gehalten das die Farbe des zugehörigen Bildes absorbierte, die des andern ungehindert durchliess. Das Resultat war ein Körper mit schwarzen Konturen auf einen Hintergrunde dessen Farbe durch die Janinsche Mischfarbe der beiden Absorptiongläser gegeben war." This method, though used, and possibly reinvented by D'Almeida, appears to have attracted little attention, and to have received few applications until, during the years 1891-1895, Duhauron brought out and patented his "Estampes, photographies et tableaux stéréoscopiques produisant leur effet en plein jour sans l'aide du stéréoscope." We do not remember the year our attention was first drawn to the examples of Duhauron's method, brought out by the Deutscher Verlag, Berlin, as *Plastographical Views of the World*, and sold by A. E. Sparrow, 7 High Holborn, from whom, we believe, they can still be obtained, but we did not find them so well known to the visitors to the Exhibition as might have been expected. Naturally, after the publication of these views, the idea of applying the method to the representation of geometrical figures occurred to several amateurs, especially to those interested in stereoscopic work, notably to Mr. F. G. Smith, whose collection, by a curious coincidence, was also shown at the Exhibition. But the difficulty of the drawings, which require something more than the ordinary perspective of the schools, prevented any general use of the method. The mathematical world, and especially that part of it concerned in teaching, are, therefore, greatly indebted both to M. H. Richard and M. H. Vuibert for designing, executing, and exhibiting the magnificent collection displayed at Cambridge. We were glad to learn that a selection of the diagrams shown was about to be published, and looked forward with great interest to the arrival of the brochure containing it. This contains about 16 pages of text descriptive of the method and its applications, and 16 pages of "Anaglyphes." These are selected so as to show the varied possibilities of the method as applied :—(1) To ordinary school solid geometry, (2) to spherical geometry, descriptive geometry, crystallography, and to physics.

We commend especially the representations of a Poinsot star-dodecahedron (p. 23), the hexagonal section of a cuboid (p. 2), the crystals (pp. 29, 30), A pair of colour screens accompanies each copy. The very low price (1s. 6d.) and the beauty of the diagrams should ensure a wide circulation. The author and

publisher propose to issue separate treatises on various branches. We shall welcome them cordially, and especially look forward to the appearance of the volume which is to be devoted to Geometry. EDWARD M. LANGLEY.

Non-Euclidean Geometry: A critical and historical study of its development. By ROBERTO BONOLA. Authorised English translation with additional appendices by H. S. Carslaw, Professor in the University of Sydney, N.S.W.; with an introduction by Federigo Enriques. Pp. xii+268. \$2 net. 1912. (Chicago: The Open Court Publishing Co.)

Bonola's *Non-Euclidean Geometry* is one of the best historical and critical studies on mathematics that have been written. Based upon an article of 1900, it has, chiefly through the German translation of 1908, become a standard work, not the least of the merits of which is its elementary character—only in the last chapter is a knowledge of more advanced mathematics required.

In this translation Prof. Carslaw has made use of some changes made in the German translation and of much new matter supplied by Prof. Bonola, and has added many notes—put, as they should be, in square brackets—and an appendix (V.) containing an elementary proof of the impossibility of proving the parallel postulate.

Chapter I. deals with the attempts to prove Euclid's parallel postulate; Chapter II. with the researches of Saccheri, Lambert, the French geometers towards the end of the eighteenth century, Legendre, W. Bolyai, Wachter, and Thibaut; Chapter III. with those of Gauss, Schweikart, and Taurinus; Chapter IV. with those of Lobachevski and J. Bolyai; and Chapter V. with the principles of the work of Riemann and Helmholtz, and with the projective theory of metrics. Appendix I. is on the fundamental principles of statics and Euclid's postulate; Appendix II. is on the problem of Clifford and Klein; Appendix III. is on the non-Euclidean parallel constructions and other allied constructions; and Appendix IV. is on the independence of projective geometry of Euclid's postulate.

The most notable of the additions made by Prof. Carslaw are the fifth appendix already referred to and part of the section on Gauss' work (see p. 67). There are a few blemishes which arise from the too literal use of the German as guide: part of p. 3 reads as if Geminus, and not Bonola, quoted from Proclus; J. Bolyai's first name is often stated to be "Johann," whereas it was "János"—or, in English, "John"; and the German equivalent of Lobachevski's name is used. Note 4 on p. 52 and note 1 on p. 53 are to be interchanged; and, in the note on p. 149, "Berichte" should be "Nachrichten."

This book should be particularly useful to teachers of elementary geometry.

Nichteuklidische Geometrie. By Dr. HEINRICH LIEBMANN. 2nd edition revised. Sammlung Schubert, Bd. XLIX. Pp. vi+222. 6.80 marks. 1912. (Göschen, Leipzig.)

The first edition of this work appeared in 1905. Prof. Liebmann is the German translator of Bonola's book, and the book under review may very well be read after Bonola. It is more of a text-book, history is hardly touched upon, and it is more advanced. But yet the style is condensed, clear, and forcible. It is not necessary to do more than refer to the additions to this edition. Often the axiomatic considerations of Hilbert and others are introduced; some researches of Study's are introduced; and—what is especially interesting at the present time—the elements of the theory of relativity and its connection with hyperbolic geometry are discussed.

It is disappointing to find no reference to the work of the Italian mathematicians—especially Peano and Pieri—which, for a great part, is previous to the work of Hilbert, and deals, in a logically far superior manner, with much the same class of questions.

La Logique Déductive dans sa dernière Phase de Développement. By ALESSANDRO PADOA. With a preface by Giuseppe Peano. Pp. 106. 1912. (Gauthier-Villars.)

This admirable exposition of the mathematical logic of Peano and his school was given in the form of lectures delivered under the auspices of the University

of Geneva, and was published in the *Revue de Métaphysique et de Morale* for 1912. It fulfils this purpose remarkably well. Modern logicians would hardly assent to the description of it as depicting the latest phase of development of deductive logic, since there is, for example, no notice taken of the work of Frege on such questions as definition without hypotheses and of Russell on the theory of logical types. Indeed, Frege is not mentioned, even in the Bibliography (p. 13)—in which, by the way, might be mentioned Jevons and Venn as well; and, while Russell is mentioned, it is stated (p. 61) that Russell's logic of relations is reducible to Peano's logic—a statement which cannot be admitted, but which has, fortunately, no ill effect on the rest of the book. The author, like Peano, does not distinguish clearly enough propositions and propositional functions (cf., *e.g.*, p. 42). But otherwise the book is accurate and very readable.

PHILIP E. B. JOURDAIN.

Éléments de la Théorie des groupes de Substitutions. By J. A. DE SÉGUIER. Pp. x + 238. 1912. (Paris, Gauthier-Villars.)

This book is written as a sequel to the author's former *Éléments de la Théorie des Groupes Abstraits*, published in 1904.

The theory of linear homogeneous substitutions has made much progress of late. To this, however, Dr. Séguier only devotes some ten pages, mainly occupied with the reduction of a linear substitution to canonical form, about forty-five pages giving an account of Dickson's Galois Field theory, and a note on matrices at the end.

The remainder of the book is concerned with what are now often called "permutations" (instead of "substitutions"), including a note of twenty pages devoted to the application of permutations to the Theory of Equations. The theory of permutation-groups is exhaustively treated with great ability, and Dr. Séguier's book must be for some time classical in this part of the subject. He writes as an expert for experts; and it must be confessed that his presentation will be quite unintelligible to anyone who has not already a grasp of group-theory comparable with the author's. Even the veteran in the subject will hardly find the book easy reading. There are but few examples, the style is far too condensed, the text runs on continuously without the phrasing common in mathematical books, and the proofs given are not always the simplest available.

Dr. Séguier has rather a fondness for new nomenclature; many of his terms seem convenient, but his definition of a "Monomial" substitution on p. 6 seems unusual.

Ten pages full of misprints in the *Éléments de la Théorie des Groupes Abstraits* are given. It is to be hoped that the present treatise has fewer inaccuracies; at any rate only one page of corrections is here given.

HAROLD HILTON.

(1) **Higher Algebra.** By CHARLES DAVISON, Sc.D. 6s. 1912. (Cambridge University Press.)

(2) **A New Algebra.** By S. BARNARD, M.A., and J. M. CHILD, B.A., B.Sc. Vol. II. Parts IV.-VI. 4s. 1912. (Macmillan.)

(3) **An Introduction to the Infinitesimal Calculus.** By H. S. CARSLAW, Sc.D. Second edition. 5s. net. 1912. (Longmans, Green & Co.)

(4) **The Calculus for Beginners.** By W. M. BAKER, M.A. 3s. 1912. (G. Bell & Sons.)

(5) **Elements of the Differential and Integral Calculus.** By W. A. GRANVILLE, Ph.D. Revised edition. 10s. 6d. 1911. (Ginn & Co.)

These five books differ widely in scope and still more widely in merit. At the same time they have something in common. They all profess to be "elementary," and they are all concerned, at one point or another, with some of the fundamental notions of analysis.

(1) The publication of this book by the Cambridge Press can only be attributed to reprehensible carelessness on the part of its expert advisers.

A reviewer who receives a book on "Higher Algebra" naturally turns first to the chapters on limits, convergence of series, and the exponential, logarithmic, and binomial series. Dr. Davison's treatment of all these subjects can only be described as hopelessly uneducated. He shows no kind of conception of the

logical relations between different parts of the theory, and his definitions and proofs are not only extraordinarily slovenly, but are full of the grossest blunders. It would be waste of time to justify these remarks by a large number of criticisms of detail, but I may give a few illustrations from Chapter V. Dr. Davison defines a *series* in § 64, and a *convergent series* in § 65, both wrongly. As he never defines a *limit* at all, and postpones to § 180 any sort of explanation of what he means by a limit, this is only natural. The most important theorems in the chapter are contained in §§ 68, 73, 75, 78, and 79. In these sections Dr. Davison merely repeats the traditional blunders of English text-books of twenty years ago. In the first three cases, for example, he assumes what he professes to prove.

Nor can I say honestly that I think that Dr. Davison is very much happier when he gets away from the difficulties of limits and convergence. He is completely mistaken, for instance, in supposing that, in § 28, he has established the possibility of expressing a given rational function as a sum of partial fractions. Chapter IV., on "Complex Quantity," is a morass of confusion. I am quite unable to disentangle what the author regards as definition and what as proof. The proof (§ 164) that the arithmetic mean is greater than the geometric is unsound, as has been pointed out by a previous writer in the *Gazette*.* Finally, in § 170, Dr. Davison proves that "every rational integral equation of the nth degree has n, and only n, roots" without a word of explanation that he is assuming the existence of at least one root.

I have selected these examples more or less at random. The book is, in my opinion, a thoroughly bad one, which ought never to have been published. The fact that it appears under the *aegis* of a University Press leads me to think that I should say so with more emphasis than I should have otherwise considered necessary.

(2) Messrs. Barnard and Child's *New Algebra* seems to me a book of an altogether higher class than any other Algebra for schools that I have seen. That such a book should be produced by two authors with a wide experience of elementary teaching is a most encouraging sign of the times, and one particularly gratifying to the professional mathematicians who have protested against the superstition that accuracy is necessarily repellent and that slip-shod half-truths are all that can be interesting or intelligible to beginners.

The authors understand what is meant by a function, or a limit. They can distinguish between a quantity and a number, between a rational and an irrational number, between a rational number and a rational function, or between a limit and a value. Their standard of accuracy is a good deal higher than that of such a well-known book as Chrystal's *Algebra*; they give a satisfactory discussion, for example, of the infinite geometric series. The gulf which separates this book from the ordinary school text-book may easily be imagined.

In spite, or rather perhaps in consequence of this, the book is bright and interesting throughout, and very seldom difficult, and it has the inestimable merit, rare indeed in a text-book, of being written in clear and decent English. The examples are numerous and well selected. In short, I feel that I can hardly recommend it too strongly to teachers of mathematics.

I have noticed very little in the way of error or obscurity. I do not like the explanation (p. 593) of what is meant by saying that "x is large" or "x is small." To say that "x is large" means *nothing*. *Statements containing* "x is large," on the other hand, may mean something. The phrase is, to use Mr. Russell's language, "incomplete": it is not a constituent of the propositions in which it occurs. I do not suggest that the authors are guilty of any real mistake, but their language seems to me confusing. My only other criticism concerns the last page of all. Here the authors are referring forward to the proof of a theorem to be given in Part VII. Exactly how much they propose to prove there, naturally I cannot say. But the proposition which they quote, regarding the rearrangement of an absolutely convergent series, does not, as ordinarily enunciated and proved, suffice for the application they propose to make of it. Here it is necessary to rearrange an absolutely convergent *double* series. Such a problem may, it is true, be reduced to a problem of a rearrangement of a simple series; but it is a rearrangement of a different and more complicated type than that contemplated in the theorem which they seem to have in mind.

* Mr. Muirhead: *vide Math. Gazette*, vol. ii. p. 283.

(3) I have already reviewed Prof. Carslaw's excellent "Introduction" in the *Gazette* (vol. iii. p. 274). It is pleasant to find that it should have reached a second edition so quickly.

In the original edition the treatment of the logarithmic and exponential functions was open to criticism. That given in this edition is a great improvement. In spite of the authority of Prof. Carslaw and Prof. Love, I am still doubtful as to the advisability of making so much depend upon the limit of $\left(1+\frac{1}{n}\right)^n$, in an elementary book in which an adequate proof of the existence of the limit is out of the question. But I am prepared to believe that this method is found as clear and satisfactory as any other. I wish, however, that in the Appendix, where Prof. Carslaw now places the "older proofs of the theorems regarding the differentiation of e^x and $\log x$," he had explained more clearly exactly where the difficulties of these "older proofs" lie.

(4) I am unable to commend this book on any grounds, or to understand why it should have any prospect of competing with much better books already on the market. It is not attractively written nor, so far as I can see, particularly "practical," and the author's knowledge of the theory may be estimated in ten minutes by any competent critic. Such a critic I would refer in particular to the first four pages, to the discussion of the differentiation of x^n (pp. 9-10), to the treatment of differentials (p. 68), or that of areas (pp. 88-92).

Mr. Baker confines himself for the most part to the reproduction of other people's mistakes, but occasionally indulges in the expression of his own opinions, as when he defines an "independent variable" as "a quantity to which we may assign any value" (p. 1), or says that the differential coefficient "always exists in functions of every kind" (p. 6).

(5) "In this revised edition of Granville's *Calculus* the latest and best methods are exhibited . . . Those features of the first edition which contributed so much to its usefulness and popularity have been retained . . ." The author certainly does not err on the side of bashfulness, and invites a reviewer to judge him by the severest standards. Still, the book, if sometimes a little disappointing, after what the preface has led us to expect, is on the whole quite a good one.

Dr. Granville does not seem to have any very consistent standard as to what may reasonably be regarded as elementary and what not. Thus, on p. 215 he quotes the fundamental theorem that a monotonic sequence tends to a limit or to infinity. The proof he regards as beyond his range. He is thus unable to establish the existence of the exponential limit. He does not even prove that $x^n \to 0$ if $|x| < 1$; but this is apparently because he has not seen that any proof is needed. It is only natural, in the circumstances, that his treatment of series should be sketchy and inadequate. I may add that Theorem III. on p. 215, which asserts that the condition "$\lim_{n\to\infty} (S_{n+p} - S_n) = 0$, for all values of the integer p is *sufficient* to ensure the existence of a limit for S_n," is untrue (see Bromwich, *Infinite Series*, p. 46). Dr. Granville professes to be quoting Osgood's *Introduction to Infinite Series*, which I have not at hand; but I cannot believe that he is quoting correctly.

Another part of the book which is unsatisfactory, because the foundations have not been properly laid, is that which deals with integration as summation, the Fundamental Theorem of the Integral Calculus, and so on. On the other hand, formal proofs are sometimes given quite as difficult as those of more fundamental theorems that are omitted: I may instance that of the reversibility of two partial differentiations.

There is too much formal "bookwork." The reader is asked to regard the formula

$$\frac{d}{dx}(\text{arc}\csc v) = -\frac{1}{v\sqrt{(v^2-1)}}\frac{dv}{dx}$$

as such, to "memorise" it, and to "be able to state the corresponding rule in words"! The examples are numerous, but on the whole dull and lacking in variety.

I could make many other criticisms of detail; but I do not wish to appear ungenerous to a book which, while hardly likely to excite enthusiasm, has solid

merits, and is on the whole clear, readable, and reasonably accurate. To profess that I regard its methods as the "latest and best" would, however, be an exaggeration. G. H. HARDY.

The Method of Archimedes, recently discovered by Heiberg. A Supplement to **The Works of Archimedes.** 1897. Edited by SIR THOMAS L. HEATH, K.C.B. Pp. 51. 2s. 6d. net. 1912. (Cambridge University Press.)

The story of the discovery of this MS. by Heiberg has been told in *Hermes*, xlii., and in *Bibliotheca*, viii. The particular interest attaching to "The Method" has been most felicitously set forth by the Editor: "Nothing is more characteristic of the classical works of the great geometers of Greece, or more tantalising, than the absence of any indication of the steps by which they worked their way to the discovery of their great theorems. As they have come down to us, these theorems are finished masterpieces, which leave no traces of any rough-hewn stage, no hint of the manner by which they were evolved. We cannot but suppose that the Greeks had some method or methods of analysis hardly less powerful than those of modern analysis; yet, in general, they seem to have taken pains to clear away all traces of the machinery used and all the litter, so to speak, resulting from tentative efforts, before they permitted themselves to publish, in sequence carefully thought out, and with definitely and rigorously scientific proofs, the results obtained." A partial exception is now furnished by the Method, for here we have a sort of lifting of the veil, a glimpse of the interior of Archimedes' workshop as it were. Assuming the principle of the lever, he attacks geometrical problems through the medium of mechanics. For instance, he has already, in the Quadrature of the Parabola, proved the theorem that the area of a parabolic segment ABC is 4/3 of the triangle ABC. The mechanical discussion of the Method he regards, not as a demonstration, but simply as giving reasons for suspecting that the property is true—"argument has given a sort of indication." It is interesting to find that Archimedes discovered his formula for the volume of the sphere before he found an expression for the area, and that Eudoxus was the first to discover "that the volumes of a pyramid and a cone are one-third of the volumes of a prism and a cylinder respectively which have the same base and equal height," Democritus having asserted it previously but without proof. The admirable introduction by Sir T. L. Heath is a model of editorial craftsmanship.

THE LIBRARY.

THE Library has now a home in the rooms of the Teachers' Guild, 74 Gower Street, W.C. A catalogue has been issued to members containing the list of books, etc., belonging to the Association and the regulations under which they may be inspected or borrowed.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

Wanted by purchase or exchange:

1 or 2 copies of *Gazette* No. 2 (very important).
2 or 3 copies of Annual Report No. 11 (very important).
1 or 2 ,, ,, Nos. 10, 12 (very important).
1 copy ,, Nos. 1, 2.

GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE AND CO. LTD.

fess
an
DY.

Sup-
L.
)
lii.,
has
istic
han
the
ems
hint
the
nose
way
rom
are-
the
nere
ork-
rical
the
seg-
d he
that
ting
fore
over
of a
ht,"
able

wer
list
nich

any